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SENSITIVITY CONSIDERATIONS IN ADAPTIVE BEAMFORMING, (U)
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NATO ADVANCED STUDY INSTITUTE
ON SIGNAL PROCESSING
with particular reference to
Underwater Acoustics

21st August - 1st September, 1972
University of Technology
Loughborough
U.K.

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I. INTRODUCTION

The theory of optimum arrays became widely known in the underwater acoustics community through the work of Bryn [8] and Mermoz [17]. The relationship of their work to other detection and estimation problems has been discussed in [7]. Optimum array processing structures use detailed information about the signal and noise fields. Since this information is not known precisely in advance, one is led naturally to adaptive beamformers which continually adjust their parameters based upon on-line measurements of some kind. Since adaptive processors are continually adjusting, it is natural to question how sensitive performance is to small variations of the signal field, noise field and system parameters from their assumed or estimated values.

The question of sensitivities has been examined in the past ~~{6, 12, 22, 24}~~ in conjunction with "super-directive" arrays. An attempt will be made to point out the relationship of the results of this paper to those earlier results. The emphasis in this paper is on receiving arrays.

In Section II an introduction to the problem is provided using an intuitive approach. The performance measures of array gain and output power are used.

Section III presents the principal sensitivity results. The approach is to take partial derivatives of the gain and output signal power with respect to the size of signal, noise and steering perturbations.

Section IV discusses the problem of signal suppression which arises in passive adaptive processors when measurements of signal-plus-noise are used when noise only measurements are desired. Interference rejection is also discussed.

A number of optimization problems are discussed in Section V. Particular emphasis is given to formulations which somehow take sensitivity into account. It is pointed out that the simple beamformer structure is no longer optimum in most of these

problems. A more general array structure is suggested which possesses optimality properties.

Vector-matrix notation similar to that in [7] will be used throughout this paper. Column vectors, i.e., matrices with only one column are designated by underlined lower case letters such as $\underline{k}$ and $\underline{m}$. Other matrices are designated by underlined upper case letters such as $\underline{P}$, $\underline{Q}$ and $\underline{R}$. The asterisk is used to denote complex conjugate transposition. Thus, $(\underline{k}^*\underline{k})$ is a scalar and $(\underline{m}\,\underline{m}^*)$ is a dyadic square matrix.

II. PRELIMINARY DISCUSSION

The receiving array consists of an arbitrary arrangement of M sensors. When a signal is present the waveform at the output of the i-th sensor is

$$x_i(t) = v_i(t) + n_i(t)$$

where $v_i(t)$ is the signal component and $n_i(t)$ is the noise component. The noise component includes both the effects of the external noise field and internal electronic noise. The outputs of all M sensors may be represented by the following vector equation:

$$\underline{x}(t) = \underline{v}(t) + \underline{n}(t) \tag{1}$$

The detection problem is usually formulated as deciding whether or not the signal component $\underline{v}(t)$ is present. Estimation problems arise when one seeks to obtain information about particular parameters of $\underline{v}(t)$. The relationships among a number of detection and estimation problems are discussed in [7]. There it is shown that the same beamforming processor is optimum for a variety of different detection and estimation problems.

The signal vector $\underline{v}(t)$ and the noise vector $\underline{n}(t)$ are assumed to have mean values of zero and to be statistically independent of each other. Because the analysis of array processors is sometimes simplified by working in the frequency domain, it is convenient to work with the cross-spectral density matrices of $\underline{v}(t)$, $\underline{n}(t)$ and $\underline{x}(t)$ which are denoted $\underline{\tilde{P}}(\omega)$, $\underline{\tilde{Q}}(\omega)$ and $\underline{R}(\omega)$ respectively. Then

$$\underline{R}(\omega) = \underline{\tilde{P}}(\omega) + \underline{\tilde{Q}}(\omega) \tag{2}$$

when the signal is present. $\underline{\tilde{P}}(\omega)$ and $\underline{\tilde{Q}}(\omega)$ may be written as

$$\underline{\tilde{P}}(\omega) = \sigma_s^2(\omega) \quad \underline{P}(\omega) \tag{3}$$

$$\underline{\tilde{Q}}(\omega) = \sigma_n^2(\omega) \quad \underline{Q}(\omega) \tag{4}$$

where $\sigma_n^2(\omega)$ is the noise power spectral density averaged over the M sensors so that

$$\sigma_n^2(\omega) = \text{trace}\, \underline{\tilde{Q}}(\omega)/M \tag{5}$$

and $\sigma_s^2(\omega)$ is similarly defined as the signal power (or energy) spectral density averaged

over the M sensors. Thus, $\underline{P}(\omega)$ and $\underline{Q}(\omega)$ are normalized to have their traces equal to M, the number of sensors in the array. The quantity $\sigma_s^2(\omega)/\sigma_n^2(\omega)$ is the input signal-to-noise spectral ratio $(S/N)_{in}$.

A device which has the configuration shown in Figure 1 will be called a beamformer. In general, it consists of a set of filters, one for each sensor, followed by a summation device. The transfer function of the filter on the i-th sensor is $k_i^*(\omega)$. Two quantities of prime interest are the output spectral density

$$z(\omega) = \underline{k}^*(\omega)\ \underline{R}(\omega)\ \underline{k}\ (\omega) \tag{6}$$

and the array gain

$$G(\omega) = \frac{\underline{k}^*(\omega)\ \underline{P}(\omega)\ \underline{k}(\omega)}{\underline{k}^*(\omega)\ \underline{Q}(\omega)\,\underline{k}(\omega)} \tag{7}$$

where, from Figure 1, the steering vector $\underline{k}^*(\omega)$ is the row vector of filter transfer functions $\{k_1^*(\omega), \cdots, k_M^*(\omega)\}$. The sensitivity analysis will be concerned with determining the effects on $z(\omega)$ and $G(\omega)$ of varying $\underline{P}(\omega)$, $\underline{Q}(\omega)$ and $\underline{k}(\omega)$ from their nominal values.

The array gain given by (7) is the ratio of the output signal-to-noise spectral ratio to the input signal-to-noise spectral ratio. It is easily seen to be invariant if $\underline{k}^*(\omega)$ is multiplied by a scale factor. In considering sensitivity problems it is sometimes useful to normalize both the numerator and denominator of (7) by dividing by the magnitude squared of $\underline{k}^*(\omega)$. Then (7) becomes

$$G(\omega) = \frac{\underline{k}^*(\omega)\,\underline{P}(\omega)\,\underline{k}(\omega) / (\underline{k}^*(\omega)\,\underline{k}(\omega))}{\underline{k}^*(\omega)\,\underline{Q}(\omega)\,\underline{k}(\omega) / (\underline{k}^*(\omega)\,\underline{k}(\omega))} \tag{8}$$

The numerator of (8) is the normalized signal response and the denominator is the normalized noise response. We shall see that the numerator and denominator of (8) individually or more specifically their reciprocals are of fundamental importance in sensitivity analysis.

One of the reasons for working in the frequency domain is that the signal vector $\underline{v}(t)$ is usually assumed to be related to a scalar signal $s(t)$ at some source point in space by an equation of the following form:

$$\underline{v}(t) = \int \underline{m}(t-\tau)\, s(\tau)\, d\tau \tag{9}$$

where $m_i(t)$ accounts for the propagation from the source to the i-th sensor and the response of the i-th sensor itself. In the ideal case of non-dispersive propagation and distortion-free sensors, $m_i(t)$ is a simple time delay $\delta(t - T_i)$. Whenever $\underline{v}(t)$ is related to a scalar $s(t)$ by a known transformation $\underline{m}(t)$ as in (9) $\underline{P}(\omega)$ is simply the dyadic

$$\underline{P}(\omega) = \underline{m}(\omega)\,\underline{m}^*(\omega) \tag{10}$$

where $\underline{m}(\omega)$ is the Fourier transform of $\underline{m}(t)$ normalized so that $\underline{m}^*(\omega)\underline{m}(\omega) = M$.

The noise field may be the sum of a number of components. Some components of particular significance are spatially white noise, isotropic noise, and point source noise.

Spatially white noise is uncorrelated from sensor to sensor and of equal intensity at each sensor. It has the identity matrix for its normalized cross-spectral density matrix. The subscript w on $\underline{Q}_w = \underline{I}$ will be used to denote this type of noise component.

Another physically important noise component is isotropic noise in which the spatial density of the noise from all directions is the same. Since there is no preferred direction, the elements of the isotropic noise cross-spectral density matrix are all real. In a spherically isotropic field the cross-spectral density between two sensors separated by a distance d is $\sin(\omega d/c)/(\omega d/c)$ where c is the velocity of propagation. The symbol $\underline{Q}_o$ will be used to denote this isotropic noise cross-spectral density matrix. Notice that $\underline{Q}_o$ depends only on the geometry of the array and frequency. Another isotropic noise field [14] is $J_o(\omega d/c)$ which arises in two dimensional problems when the field is cylindrically isotropic.

Point source noise comes from a single point source and hence is signal-like in its spatial characteristics.

The symbol

$$\underline{Q}_d = \underline{d}(\omega)\;\underline{d}^*(\omega) \tag{11}$$

will be used to denote this type of noise component. Again the normalization $\underline{d}^*\underline{d} = M$ is assumed. Isotropic noise is simply the limit of summing infinitesimal independent noise components of equal intensity from all directions.

Henceforth, in order to simplify the notation, the dependence of various quantities on frequency ω will usually not be shown explicitly.*

When the signal matrix has the simple form of (10) the expression for the array gain becomes

$$G = \frac{|\underline{k}^*\underline{m}|^2}{\underline{k}^*\underline{Q}\,\underline{k}} \tag{12}$$

* Many of the results expressed in general vector-matrix form may also be used directly in time-domain formulation of the problem [7].

The numerator of (12) is the magnitude squared of the inner product of the steering vector $\underline{k}$ and the signal direction vector $\underline{m}$. We may think geometrically in terms of a generalized angle γ between these vectors defined as follows:

$$\cos^2\gamma = \frac{|\underline{k}^*\underline{m}|^2}{(\underline{k}^*\underline{k})(\underline{m}^*\underline{m})} \tag{13}$$

Substituting from (13) into (8) and using the relationship $\underline{m}^*\underline{m} = M$ yields

$$G = M\cos^2\gamma \left\{ \underline{k}^*\underline{k} \big/ (\underline{k}^*\ \underline{Q}\ \underline{k}) \right\} \tag{14}$$

The array gain against spatially white noise is therefore

$$G_w = M\cos^2\gamma \tag{15}$$

which is also the normalized signal response (numerator of (8)). It is a function of the misalignment of the steering vector $\underline{k}$ and the signal direction vector $\underline{m}$.

In a conventional beamformer $\underline{k}$ is chosen to be proportional to $\underline{m}$. This choice of "matching to the signal direction" makes γ equal to zero. This maximizes the gain against spatially white noise, alias normalized signal response. The maximum value of G_w in (15) is M. If G_w is less than unity the array becomes more sensitive than a single sensor to spatially white noise.

The factor $\left\{ \underline{k}^*\underline{k} \big/ (\underline{k}^*\ \underline{Q}\ \underline{k}) \right\}$ is the reciprocal of the normalized noise response (the denominator of (8)). It is also the gain ratio G/G_w which we shall denote by ρ. When the noise is spatially white this factor is unity. For an arbitrary noise matrix this gain ratio ρ is bounded as follows:

$$\frac{1}{\lambda_{max}} \leq \frac{\underline{k}^*\underline{k}}{\underline{k}^*\ \underline{Q}\ \underline{k}} \leq \frac{1}{\lambda_{min}} \tag{16}$$

where λ_{min} and λ_{max} are respectively the smallest and largest eigenvalues of $\underline{Q}$. Since $\underline{Q}$ is normalized to have its trace equal to M, the average size of the eigenvalues of $\underline{Q}$ is unity.

When the noise is isotropic the array gain

$$G_o = M\cos^2\gamma \left\{ \underline{k}^*\underline{k} \big/ (\underline{k}\ \underline{Q}_o\underline{k}) \right\} \tag{17}$$

is known as the directivity index. The gain ratio

$$G_o/G_w = \underline{k}^*\underline{k} \big/ (\underline{k}^*\ \underline{Q}_o\underline{k}) \tag{18}$$

is known as the Q-factor [15, 24] or super-gain ratio [22]. We shall call ρ the generalized super gain ratio since it involves replacing the isotropic noise matrix in (18) with a general noise matrix. The isotropic noise covariance matrix depends on the array geometry. For line arrays it becomes the identity matrix when elements are spaced at one-half

wavelength intervals. When sensors are moved closer together than one-half wavelength some of the eigenvalues of the isotropic matrix become extremely small and ρ may become large.

"Optimum" and "adaptive" beamformers choose $\underline{k}$ to be something other than proportional to $\underline{m}$. The underlying philosophy is to accept a reduction in normalized signal response in order to achieve an even greater reduction in normalized noise response, thereby improving the gain. Since normalized signal response and gain against spatially white noise are the same, these processors can become very sensitive to white noise if too great a reduction in normalized signal response takes place. An "optimum" steering vector $\underline{k}^*$ under a number of criteria [7] is

$$\underline{k}^* = \underline{m}^*\underline{Q}^{-1} \tag{19}$$

Lewis and Schultheiss [14] present an interesting discussion of the gain of optimum and conventional processors and their relationship to the eigenvalues of the noise covariance matrix. For the choice of $\underline{k}$ given by (19) the gain, white noise gain and generalized super-gain ratio become respectively

$$G = \underline{m}^*\underline{Q}^{-1}\underline{m} \tag{20}$$

$$G_w = \frac{(\underline{m}^*\underline{Q}^{-1}\underline{m})^2}{\underline{m}^*\underline{Q}^{-2}\underline{m}} \tag{21}$$

$$\rho = \frac{G}{G_w} = \frac{\underline{m}^*\underline{Q}^{-2}\underline{m}}{\underline{m}^*\underline{Q}^{-1}\underline{m}} \tag{22}$$

The ratio (22) of course still satisfies the bound given by (16). McDonough [16] presents a complicated expression relating (21) to the ratio of the largest and smallest eigenvalues of $\underline{Q}$. He concludes (21) may become very small if $\lambda_{max}/\lambda_{min}$ is large.

The theory of super-directive arrays involves achieving a high directivity index by using the optimum steering vector for isotropic noise $\underline{k}^* = \underline{m}^*\underline{Q}_o^{-1}$. The literature on super-directive arrays is extensive [6, 12, 22, 23, 24, 25]. It is well known that high directivity index (nearly equal to M^2) can be achieved with endfire steering of arrays of closely spaced elements. In order to achieve these high gains the steering vector is made almost orthogonal to the signal direction vector and the gain against white noise becomes extremely small. For example, Gilbert and Morgan [12] report a maximum directivity index of 15.8 but a corresponding white noise gain of only 1.5×10^{-4} with a four element endfire array with 1/16 wavelength inter-element spacing. They also present the result that the maximum directivity index of an array averaged over all steering directions is

equal to the number of elements M. To see this, we write (20) for isotropic noise as

$$G = \text{trace}\,(\underline{Q}_o^{-1}\,\underline{m}\,\underline{m}^*) \tag{23}$$

and notice that the average of $\underline{m}\,\underline{m}^*$ over all directions is by definition the isotropic matrix $\underline{Q}_o$.

Sensitivity analyses of super-directive arrays have found the super-gain ratio and the reciprocal of the white noise gain to be useful measures of sensitivity.

In this paper we shall find that $1/G_w$ and ρ play key roles in sensitivity whenever the perturbations are statistically independent from element to element.

The causes of variations of $\underline{P}$, $\underline{Q}$ and $\underline{k}$ from their nominal values can be many and varied. They include: violations of a priori assumptions such as those concerning plane wave or point source nature of signal or interference, or isotropic nature of the noise field; imperfect measuring devices and finite measuring intervals used to estimate $\underline{Q}$; non-stationarity of the noise field; presence of signal in noise estimates in passive systems; position errors in sensor locations; amplitude and phase errors in analog components; and sampling and quantization errors in digital components. In adaptive beamformers the nature of these variations is usually related to the inter-play between a priori assumptions and on-line measurements. Sensitivity to variations in $\underline{k}$ are especially important since the underlying philosophy of adaptive beamforming is continuing adjustment of the steering vector.

III. GENERAL SENSITIVITY RESULTS

1. Noise Perturbations

Suppose that the noise field is made up of several components so that $\underline{Q}$ may be expressed as a sum of Hermitian matrices of the form

$$\underline{Q} = \alpha_1\underline{Q}_1 + \alpha_2\underline{Q}_2 + \cdots\cdots + \alpha_n\underline{Q}_n \tag{24}$$

where

$$\alpha_i \geq 0,\ i = 1, \cdots, n \qquad \sum_{i=1}^{n} \alpha_i = 1$$

Each $\underline{Q}_i$ in (24) is non-negative definite and has its trace equal to M so that it is a legitimate normalized noise cross-spectral matrix. The parameter α_i is the relative strength of the i-th component. The decomposition of $\underline{Q}$ into components is not unique but is sometimes useful when various physical sources of noise exist. When $\underline{Q}$ is given in the form of (24) the array gain (7) may be expressed as follows:

$$G = \left(\sum_{i=1}^{n} \alpha_i/G_i\right)^{-1} \tag{25}$$

where G_i is the array gain against the i-th noise component; that is

$$G_i = \frac{\underline{k}^* \underline{P} \underline{k}}{\underline{k}^* \underline{Q}_i \underline{k}} \tag{26}$$

From (25) it is evident that a large value of α_i/G_i for any component will cause the gain G to be small. Thus a relatively weak noise component (small α_i) can limit the overall array gain if the gain against that particular component is very small. Hence choices of $\underline{k}$ which give low gain against a type of noise which is likely to be present result in systems which are sensitive to small amounts of that type of noise.

Similarly (6) may be rewritten as follows:

$$z = \sigma_s^2 \underline{k}^* \underline{P} \underline{k} \left[1 + \frac{\sigma_n^2}{\sigma_s^2} \sum_{i=1}^{n} \alpha_i / G_i \right] \tag{27}$$

Quantitative measures of sensitivity to changes in the noise matrix can be obtained by differentiating (25) and (27). Suppose that

$$\underline{Q} = (1 - \alpha) \underline{Q}_1 + \alpha \underline{Q}_2 \tag{28}$$

The fractional sensitivity of G and z to the substitution of a little $\underline{Q}_2$-type noise for an equal amount of $\underline{Q}_1$-type noise are

$$\left(\frac{dG/d\alpha}{G}\right)_{\alpha = o} = 1 - (G_1/G_2) \tag{29}$$

and

$$\left(\frac{dz/d\alpha}{z}\right)_{\alpha = o} = \frac{(G_1/G_2) - 1}{1 + G_1 \ \sigma_s^2 / \sigma_n^2} \tag{30}$$

or

$$\left(\frac{dz/d\alpha}{z}\right)_{\alpha = o} = \frac{-\left(\frac{dG/d\alpha}{G}\right)_{\alpha = o}}{1 + (S/N)_o} \tag{31}$$

where $(S/N)_o = (\sigma_s^2 / \sigma_n^2) G_1$ is the output signal to noise ratio. When $G_1/G_2 > 1$, the substitution of $\underline{Q}_2$-type noise for $\underline{Q}_1$-type noise causes a decrease in gain and an increase in output power. The gain ratio G_1/G_2 in (29) and (30) may be expressed as

$$\frac{G_1}{G_2} = \frac{\underline{k}^* \underline{Q}_2 \underline{k}}{\underline{k}^* \underline{Q}_1 \underline{k}} \tag{32}$$

In the special case of spatially white noise (32) reduces to the generalized super-gain ratio

$$\frac{G_1}{G_w} = \frac{\underline{k}^* \underline{k}}{\underline{k}^* \underline{Q}_1 \underline{k}} = \rho \tag{33}$$

discussed earlier. If, in addition $\underline{Q}_1$-type noise is isotropic, (33) becomes the super-gain ratio or Q-factor.

2. Signal Perturbations

Similarly we may conceive of the signal field being made up of several components so that $\underline{P}$ may be expressed as

$$\underline{P} = \sum_{i=1}^{n} \beta_i \underline{P}_i \; ; \quad \beta_i \geq 0 \quad i = 1, \cdots, n \; ; \quad \sum_{i=1}^{n} \beta_i = 1 \tag{34}$$

Each $\underline{P}_i$ is Hermitian and non-negative definite with trace equal to M.

Then

$$G = \sum_{i=1}^{n} \beta_i G_i \tag{35}$$

and

$$z = \sigma_n^2 \underline{k}^* \underline{Q} \underline{k} \left[1 + \frac{\sigma_s^2}{\sigma_n^2} \sum_{i=1}^{n} \beta_i G_i\right] \tag{36}$$

where

$$G_i = \frac{\underline{k}^* \underline{P}_i \underline{k}}{\underline{k}^* \underline{Q} \underline{k}} \tag{37}$$

is the gain for the i-th component of the signal field. The overall gain is simply the sum of the gain for each component weighted by its relative strength.

Suppose that

$$\underline{P} = (1 - \beta) \underline{P}_1 + \beta \underline{P}_2 \tag{38}$$

The fractional sensitivities of G and z to the substitution of a little $\underline{P}_2$-type signal for an equal amount of $\underline{P}_1$-type signal are

$$\left(\frac{dG/d\beta}{G}\right)_{\beta = o} = \frac{G_2}{G_1} - 1 \tag{39}$$

and

$$\left(\frac{dz/d\beta}{z}\right)_{\beta = o} = \frac{(G_2/G_1) - 1}{1 + \sigma_n^2/(G_1 \sigma_s^2)} \tag{40}$$

or

$$\left(\frac{dz/d\beta}{z}\right)_{\beta = o} = \frac{(S/N)_o}{1 + (S/N)_o} \left(\frac{dG/d\beta}{G}\right)_{\beta = o} \tag{41}$$

The gain ratio G_2/G_1 appearing in (39) and (40) may be expressed as

$$\frac{G_2}{G_1} = \frac{\underline{k}^* \underline{P}_2 \underline{k}}{\underline{k}^* \underline{P}_1 \underline{k}} \tag{42}$$

When $\underline{P}_1$ is the result of a point signal source with direction vector $\underline{m}$ and $\underline{P}_2$ is the identity matrix representing independent signal perturbations from sensor to sensor (42) becomes

$$G_2/G_1 = \underline{k}^*\underline{k} \Big/ \left|\underline{k}^*\underline{m}\right|^2 = 1/G_w \tag{43}$$

and

$$G_2 = \underline{k}^*\underline{k} \Big/ (\underline{k}^*\,\underline{Q}\,\underline{k}) = \rho \tag{44}$$

which are by now familiar expressions. Low white noise gain in (43) means large response to a spatially incoherent signal component.

Before proceeding it is useful to consider how the situation represented by (38) might arise. Suppose that the signal vector suffered random perturbations so that it could be written as

$$\underline{m} = \sqrt{1-\beta}\ \underline{m}_1 + \sqrt{\beta}\ \underline{\delta} \tag{45}$$

when $\underline{\delta}$ is a vector of random perturbations with zero mean and normalized covariance matrix $E[\underline{\delta}\underline{\delta}^*] = \underline{P}_2$. The factor $\sqrt{1-\beta}$ in (45) provides the normalization so that

$$E[\underline{m}^*\underline{m}] = \text{trace}\,[E(\underline{m}\,\underline{m}^*)] = M \tag{46}$$

Then

$$E[\left|\underline{k}^*\underline{m}\right|^2] = (1-\beta)\left|\underline{k}^*\underline{m}_1\right|^2 + \beta\,\underline{k}^*\underline{P}_2\underline{k} \tag{47}$$

In this situation, the sensitivity results (39) and (40) should be interpreted on an ensemble average basis.

From (39) it is evident that small perturbations on the average will not cause severe degradation in G or z even if $G_2 \ll G_1$. However, when $G_2 \gg G_1$ rapid increases in G and z are to be expected. Of course, any individual perturbation of the type (45) may deviate significantly from this average behavior. From (43) we see that when $\underline{k}$ is nearly orthogonal to $\underline{m}$ so that $1/G_W$ is large, random signal perturbations will on the average cause an increase in G and z by decreasing γ. Of course, unpredictable increases in G and z due to slight perturbations in $\underline{m}$ with no change in signal power can be a source of confusion and may be undesirable. This is especially true in processors which are supposedly constrained.

Another approach to the formulation of the signal sensitivity problem is to assume $\underline{P}$ is of the dyadic form of (10) and to allow amplitude and phase perturbations to occur to the individual components of $\underline{m}$, i.e.,

$$m_j = m_j^o\,(1 + a_j)\ \exp(\sqrt{-1}\ \xi_j) \tag{48}$$

when m_j^o is the nominal value of the j-th component of $\underline{m}$ and the amplitude and phase perturbations have zero mean and are assumed to be independent of each other. This approach

has been taken in a recent paper by McDonough [16] following the lead of Gilbert and Morgan [12]. When m_j is given by (48), the quantity $E(|\underline{k}^*\underline{m}|^2)$ becomes

$$E(|\underline{k}^*\underline{m}|^2) = \sum_{i,j} k_i^* k_j m_i^o m_j^{o*} C_{ij} \tag{49}$$

where

$$C_{ij} = \begin{cases} 1 + E(a_j^2), & \text{for } i = j \\ \{1 + E(a_i a_j)\} E[\exp(\sqrt{-1}(\xi_i - \xi_j))], & \text{for } i \neq j \end{cases} \tag{50}$$

Notice that the quantity $\underline{m}$ in (48) is not normalized as $\underline{m}$ was in (46) since for $\underline{m}$ defined in (48)

$$E[\underline{m}^*\underline{m}] = M + \sum_j |m_j|^2 E(a_j^2) > M \tag{51}$$

As shown by McDonough, (49) may be simplified under the following additional assumptions which were made by Gilbert and Morgan:

1. Amplitude perturbations are independent from sensor to sensor and of equal variance, i.e., $E[a_i a_j] = \sigma_a^2 \delta_{ij}$.
2. Phase perturbations are Gaussian, small, independent from sensor to sensor and of equal variance, i.e., $E[\xi_i \xi_j] = \sigma_\xi^2 \delta_{ij}$, $\sigma_\xi^2 \ll 1$.
3. m_j is a pure phasing, i.e., $|m_j|^2 = 1$, $j = 1, \cdots, n$.

Under the above assumptions

$$C_{ij} \begin{cases} = 1 + \sigma_a^2 & \text{for } i = j \\ \approx 1 - \sigma_\xi^2 & \text{for } i \neq j \end{cases} \tag{52}$$

and (49) becomes

$$E[|\underline{k}^*\underline{m}|^2] = (1 - \sigma_\xi^2)|\underline{k}^*\underline{m}^o|^2 + (\sigma_a^2 + \sigma_\xi^2)\underline{k}^*\underline{k} \tag{53}$$

which is almost identical to (47) for the corresponding case $\underline{P}_2 = \underline{I}$. The difference can be attributed to the normalization of (45).

In general (49) can be made to look like (47) by defining a matrix $\underline{F}$ with elements

$$F_{ij} = m_i^o m_j^{o*} (C_{ij} - 1) \tag{52}$$

Then (49) may be written

$$E[|\underline{k}^*\underline{m}|^2] = |\underline{k}^*\underline{m}_o|^2 + \underline{k}^*\underline{F}\,\underline{k} \tag{53}$$

Much of the sensitivity work [12, 16, 24] in the field of antenna arrays has been concerned with the quantity $E[|\underline{k}^*\underline{m}|^2]/|\underline{k}_o^*\underline{m}_o|^2$ when the components of $\underline{k}$ and/or $\underline{m}$ have been perturbed in amplitude and/or phase from their nominal values $\underline{k}_o$ and $\underline{m}_o$. From (49) it is evident that the type of result will be the same whether C_{ij} arises from

perturbations of $\underline{m}$, $\underline{k}$, or some combination of both.

3. Steering Perturbations

As suggested above there are similarities between signal perturbations and perturbations in the steering vector $\underline{k}$. Perturbations in $\underline{k}$ may also be approached in two ways, analogous to (45) and (48) respectively. However, there is an important difference in that perturbations in $\underline{k}$ affect both signal and noise terms.

Following the approach of (45), suppose that

$$\underline{k} = \sqrt{1-\epsilon}\ \underline{k}_1 + \sqrt{\epsilon}\ \underline{\eta} \tag{54}$$

where $\underline{\eta}$ is a vector of random perturbations with zero mean and covariance matrix $E[\underline{\eta}\underline{\eta}^*] = \underline{W}$ where $\underline{W}$ is normalized to have its trace equal to $\underline{k}_1^* \underline{k}_1$. Then expected output power is

$$\bar{z} = E[\underline{k}^* \underline{R}\, \underline{k}] = (1-\epsilon)\, \underline{k}_1^* \underline{R}\, \underline{k}_1 + \epsilon \operatorname{trace}(\underline{R}\,\underline{W}) \tag{55}$$

and the sensitivity of the expected output power is

$$\left(\frac{d\bar{z}/d\epsilon}{\bar{z}}\right)_{\epsilon=0} = \frac{\operatorname{trace}(\underline{R}\,\underline{W})}{\underline{k}_1^* \underline{R}\, \underline{k}_1} - 1 \tag{56}$$

Defining a gain $\tilde{\tilde{G}}$ as follows:

$$\tilde{\tilde{G}} = E(\underline{k}^* \underline{P}\, \underline{k}) / E(\underline{k}^* \underline{Q}\, \underline{k}) \tag{57}$$

we obtain

$$\tilde{\tilde{G}} = \frac{(1-\epsilon)\, \underline{k}_1^* \underline{P}\, \underline{k}_1 + \epsilon \operatorname{trace}(\underline{P}\,\underline{W})}{(1-\epsilon)\, \underline{k}_1^* \underline{Q}\, \underline{k}_1 + \epsilon \operatorname{trace}(\underline{Q}\,\underline{W})} \tag{58}$$

The sensitivity of this gain is

$$\left(\frac{d\tilde{\tilde{G}}/d\epsilon}{\tilde{\tilde{G}}}\right)_{\epsilon=0} = \frac{\operatorname{trace}(\underline{P}\,\underline{W}/G_1) - \operatorname{trace}(\underline{Q}\,\underline{W})}{\underline{k}_1^* \underline{Q}\ \underline{k}_1} \tag{59}$$

Simplifications again occur when the perturbations are independent from sensor to sensor and of equal variance so that $\underline{W} = (\underline{k}_1^* \underline{k}_1 / M)\, \underline{I}$. Then (55) becomes

$$\bar{z} = (1-\epsilon)\, \underline{k}_1^* \underline{R}\, \underline{k}_1 + \epsilon (\sigma_s^2 + \sigma_n^2)\ \underline{k}_1^* \underline{k}_1 \tag{60}$$

and (56) becomes

$$\left(\frac{d\bar{z}/d\epsilon}{\bar{z}}\right)_{\epsilon=0} = \frac{\underline{k}_1^* \underline{k}_1}{\underline{k}_1^* \underline{Q}\, \underline{k}_1} \left\{ \frac{1 + (S/N)_{in}}{1 + (S/N)_o} \right\} - 1 \tag{61}$$

When $(S/N)_{in}$ and $(S/N)_o$ are both much less than unity the output power sensitivity of (61) is approximately equal to $\rho - 1$. When $(S/N)_{in}$ and $(S/N)_o$ are both much larger than unity, it is approximately equal to $\{(1/G_w) - 1\}$.

A similar simplification occurs in (58) which becomes

$$\tilde{\tilde{G}} = \frac{(1-\epsilon)\,\underline{k}_1^*\,\underline{P}\,\underline{k}_1 + \epsilon\,\underline{k}_1^*\,\underline{k}_1}{(1-\epsilon)\,\underline{k}_1^*\,\underline{Q}\,\underline{k}_1 + \epsilon\,\underline{k}_1^*\,\underline{k}_1} \tag{62}$$

and (59) which becomes

$$\left(\frac{d\tilde{\tilde{G}}/d\epsilon}{\tilde{\tilde{G}}}\right)_{\epsilon=o} = \frac{\underline{k}_1^*\ \underline{k}_1}{\underline{k}_1^*\,\underline{Q}\,\underline{k}_1}\left\{\frac{1}{G_1} - 1\right\} \tag{63}$$

For the usual case of $G_1 \gg 1$ the gain sensitivity of (63) is approximately equal to $-\rho$.

Following the alternate approach of (48) and considering amplitude and phase perturbations, we may describe $\underline{k}$ as follows:

$$k_j = k_j^o\,(1+b_j)\exp(\sqrt{-1}\ \ \phi_j) \tag{64}$$

where k_j^o is the nominal value of the j-th component of $\underline{k}$. The amplitude perturbation b_j and the phase perturbation ϕ_j have zero mean and are assumed to be independent of each other. When $\underline{k}$ is described by (62) the expected output power is

$$\bar{z} = E(\underline{k}^*\,\underline{R}\,\underline{k}) = \sum_{i,j} k_i^{o*}\,k_j^o\,D_{ij}\,R_{ij} \tag{65}$$

where R_{ij} is the i, j-th component of $\underline{R}$ and in analogy to (50)

$$D_{ij} = \begin{cases} 1 + E(b_j^2)\,, & \text{for } i = j \\ \left\{1 + E(b_i b_j)\right\}\ E[\exp(\sqrt{-1}\ \ (\phi_j - \phi_i))]\,, & \text{for } i \neq j \end{cases} \tag{66}$$

Similarly $\tilde{\tilde{G}}$ defined in (57) becomes

$$\tilde{\tilde{G}} = \frac{\sum_{i,j} k_i^{o*}\,k_j^o\,D_{ij}\,P_{ij}}{\sum_{i,j} k_i^{o*}\,k_j^o\,D_{ij}\,Q_{ij}} \tag{67}$$

where P_{ij} and Q_{ij} are i, j-th components of $\underline{P}$ and $\underline{Q}$ respectively. Under the following assumptions (63) and (65) may be simplified considerably:

1. Amplitude perturbations are independent from sensor to sensor and of equal variance, i.e., $E[b_i b_j] = \sigma_b^2\ \delta_{ij}$.
2. Phase perturbations are Gaussian, small, independent from sensor to sensor and of equal variance, i.e., $E[\phi_i\ \phi_j] = \sigma_\phi^2\ \delta_{ij}\,,\ \sigma_\phi^2 \ll 1$.
3. Either $\left|k_i\right|^2 = \left|k_j\right|^2$ for i, j = 1, $\cdots$ M, or $Q_{ii} = P_{ii} = 1$ for i = 1, $\cdots$, M.

Then (65) becomes

$$\bar{z} = (1-\sigma_\phi^2)\,\underline{k}^{o*}\ \underline{R}\,\underline{k}^o + (\sigma_\phi^2 + \sigma_b^2)\,(\sigma_s^2 + \sigma_n^2)\,\underline{k}^{o*}\ \underline{k}^o \tag{69}$$

which closely resembles (60). Under the same assumptions (67) becomes

ASSUMPTIONS	GAIN SENSITIVITY	OUTPUT SENSITIVITY
$G = \underline{k}^* \underline{P}\, \underline{k} \big/ \underline{k}^* \underline{Q}\, \underline{k}$ $z = \underline{k}^* \underline{R}\, \underline{k}$	$\left(\frac{dG/d\epsilon}{G}\right)_{\epsilon = o}$	$\left(\frac{dz/d\epsilon}{z}\right)_{\epsilon = o}$
Noise Variations		
$\underline{Q} = (1-\epsilon)\,\underline{Q}_1 + \epsilon\, \underline{Q}_2$	$1 - (G_1/G_2)$ (29)	$[(G_1/G_2) - 1] \big/ (1 + (S/N)_o)$ (31)
For $\underline{Q}_2 = \underline{I}$	$1 - \rho$ (33)	$(\rho - 1) \big/ (1 + (S/N)_o)$
Signal Variations		
$\underline{P} = (1-\epsilon)\,\underline{P}_1 + \epsilon\, \underline{P}_2$	$(G_2/G_1) - 1$ (39)	$[(G_2/G_1) - 1]\,(S/N)_o \big/ [1 + (S/N)_o]$ (41)
For $\underline{P}_1 = \underline{m}\,\underline{m}^*$, $\underline{P}_2 = \underline{I}$	$(1/G_w) - 1$ (43)	$[(1/G_w) - 1]\,(S/N)_o \big/ [1 + (S/N)_o]$
Steering Variations	$\tilde{\tilde{G}}$ for G	$\bar{z}$ for z
$\underline{k} = (1-\epsilon)\,\underline{k}_1 + \sqrt{\epsilon}\; \underline{\eta}$ $E(\underline{\eta}\underline{\eta}^*) = \underline{W}$, trace $\underline{W} = \underline{k}^* \underline{k}$	$\dfrac{\text{trace}\,(\underline{P}\,\underline{W}/G_1) - \text{trace}\,(\underline{Q}\,\underline{W})}{\underline{k}_1^* \underline{Q}\, \underline{k}_1}$ (59)	$\dfrac{\text{trace}\,(\underline{R}\,\underline{W})}{\underline{k}_1^* \underline{R}\, \underline{k}_1} - 1$ (56)
For $\underline{W} = (\underline{k}^*\underline{k}/M)\, \underline{I}$	$-\rho \left\{ 1 - (1/G_1) \right\} \approx -\rho$ (63)	$\rho \left\{ \dfrac{1 + (S/N)_{in}}{1 + (S/N)_o} \right\} - 1$

TABLE I. SUMMARY OF SENSITIVITY RESULTS.

$$\tilde{\tilde{G}} = \frac{(1 - \sigma_\phi^2)\, \underline{k}^{o*} \underline{\underline{P}} \underline{k}^o + (\sigma_\phi^2 + \sigma_b^2)\, \underline{k}^{o*} \underline{k}^o}{(1 - \sigma_\phi^2)\, \underline{k}^{o*} \underline{\underline{Q}} \underline{k}^o + (\sigma_\phi^2 + \sigma_b^2)\, \underline{k}^{o*} \underline{k}^o} \tag{70}$$

which closely resembles (62).

Finally let us note that when $\underline{m}$ is perturbed as in (48) and $\underline{k}$ is independently perturbed as in (64),

$$\frac{E[\,|\underline{k}^* \underline{m}|^2\,]}{E[\underline{k}^* \underline{\underline{Q}}\, \underline{k}]} \approx \frac{\Sigma\, k_i^{o*} k_j^o m_i^o m_j^{o*} D_{ij} C_{ij}}{\Sigma\, k_i^{o*} k_j^o D_{ij} Q_{ij}} \tag{71}$$

4. Summary

Table I presents a summary of some of the more important sensitivity results derived in this Section.

IV. INTERFERENCE REJECTION AND SIGNAL SUPPRESSION

The problem of designing an array to be insensitive to a point source of noise has received considerable attention in the literature [1, 2, 8, 20]. In the notation of this paper, the goal is to reject a component of the noise field of the form of (11).

The gain of any beamformer against such an interference is

$$G_d = |\underline{k}^* \underline{m}|^2 \Big/ |\underline{k}^* \underline{d}|^2 = \cos^2\gamma \Big/ \cos^2\theta \tag{72}$$

where θ is the generalized angle between the steering vector $\underline{k}$ and the interference direction vector $\underline{d}$. The quantity $\cos^2\theta$ is obtained by replacing $\underline{m}$ in (13) by $\underline{d}$.

Perhaps, the conceptually simplest approach to interference rejection is to use a steering vector which is orthogonal to $\underline{d}$ so that $\cos^2\theta = 0$. Such a beamformer completely nulls out the unwanted interference. The quality of the null will be degraded if either $\underline{k}$ or $\underline{d}$ suffers random perturbations. Because of the symmetry of the expression

$$\cos^2\theta = |\underline{k}^* \underline{d}|^2 \Big/ (\underline{k}^* \underline{k})\, (\underline{d}^* \underline{d}) \tag{73}$$

perturbations in $\underline{k}$ or $\underline{d}$ have similar effects on the null. Perturbations in $\underline{k}$ also affect the signal response while perturbations in $\underline{d}$ do not. The quantity $E(|\underline{k}^* \underline{d}|^2)$, when $\underline{d}$ suffers random perturbations, may be obtained directly from the results (47) or (49) presented for the $E(|\underline{k}^* \underline{m}|^2)$, when $\underline{m}$ underwent random perturbations. In the simplest case of independent perturbations from sensor to sensor

$$E[\cos^2\theta\,] = \sigma^2/M \tag{74}$$

where σ^2 is equal to β in (47) or the combined variance of the amplitude and phase perturbations in (53). In this situation the expected gain is

$$E(G_d) = G_w/\sigma^2 \tag{75}$$

Thus if G_w is small, nulling becomes impractical.

The null-steering beamformer of Anderson [1,2] uses the following steering vector*:

$$\underline{k}^* = \underline{m}^* [\underline{I} - \underline{d}\underline{d}^*/M] \tag{76}$$

For this steering vector the white noise gain G_w is given by the following equation:

$$G_w = M[1 - |\underline{m}^*\underline{d}|^2/M^2] = M[1 - \cos^2\mu] = M\sin^2\mu \tag{77}$$

where μ is the generalized angle between the signal direction vector and the interference direction vector. Thus, the expected gain (75) against the interference will be low when the interference and signal are closely spaced such as when the interference is within the main lobe of a beam steered conventionally in the signal direction.

When an interference is added to an existing noise field the noise matrix becomes

$$\tilde{\underline{Q}} = \sigma_1^2 \underline{Q}_1 + \sigma_d^2 \underline{d}\underline{d}^* = (\sigma_1^2 + \sigma_d^2)\underline{Q} \tag{78}$$

The optimum steering vector is

$$\underline{k}^* = \underline{m}^*\underline{Q}^{-1} = [1 + (\sigma_d^2/\sigma_1^2)]\ \underline{m}^* \left[\underline{Q}_1^{-1} - \frac{\underline{Q}_1^{-1}\underline{d}\underline{d}^*\underline{Q}_1^{-1}}{\underline{d}^*\underline{Q}_1^{-1}\underline{d} + \sigma_1^2/\sigma_d^2} \right] \tag{79}$$

Then

$$\underline{k}^*\underline{d} = [1 + (\sigma_d^2/\sigma_1^2)]\ \underline{m}^*\underline{Q}_1^{-1}\underline{d} \left[1 - \frac{1}{1 + [\underline{d}^*\underline{Q}_1^{-1}\underline{d}\ \sigma_d^2/\sigma_1^2]^{-1}} \right] \tag{80}$$

and

$$\underline{k}^*\underline{m} = [1 + (\sigma_d^2/\sigma_1^2)]\ \underline{m}^*\underline{Q}_1^{-1}\underline{m} \left\{ 1 - \left[\frac{|\underline{m}^*\underline{Q}_1^{-1}\underline{d}|^2}{(\underline{m}^*\underline{Q}_1^{-1}\underline{m})(\underline{d}^*\underline{Q}_1^{-1}\underline{d})} \right] \left(\frac{1}{1 + [\underline{d}^*\underline{Q}_1^{-1}\underline{d}\ \sigma_d^2/\sigma_1^2]^{-1}} \right) \right\} \tag{81}$$

The gain of the "optimum beamformer is also given by (81).

In order to interpret (80) and (81) it is useful to consider the meaning of various quantities which appear in these equations:

σ_d^2/σ_1^2 -- interference to noise ratio at the input $(I/N)_{in}$.

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* The matrix $[\underline{I} - \underline{d}\underline{d}^*/M]$ is a projection operator which passes only the component of $\underline{m}$ which is orthogonal to $\underline{d}$.

$\underline{m}^* \underline{Q}_1^{-1} = \underline{k}_1$ -- optimum steering vector in the absence of interference.

$\underline{d}^* \underline{Q}_1^{-1} \underline{d}\ \sigma_d^2/\sigma_1^2 = (I/N)_{max}$ -- maximum possible output interference-to-noise ratio, that is, the output interference-to-noise ratio of an optimum beamformer, for $\underline{Q}_1$-type noise, steered in the direction of the interference.

$\left|\underline{m}^* \underline{Q}_1^{-1} \underline{d}\right|^2 = \left|\underline{k}_1 \underline{d}\right|^2$ -- response in the interference direction of an optimum beamformer, for $\underline{Q}_1$-type noise, steered in the signal direction.

$\underline{m}^* \underline{Q}_1^{-1} \underline{m}$ -- optimum gain in the absence of interference.

The "optimum" beamformer does a tradeoff between nulling the interference and preserving gain against $\underline{Q}_1$-type noise. This tradeoff involves the interference-to-noise ratio and the sidelobe level in the interference direction of the optimum beamformer, for $\underline{Q}_1$-type noise, steered in the signal direction. When $\underline{Q}_1$-type noise is spatially white so that $\underline{Q}_1 = \underline{I}$, the limit of the steering vector (79) for large input interference-to-noise ratios is the Anderson null-steering processor (76) to within a constant of proportionality.

When $(I/N)_{max}$ is small the interference has little effect on the optimum steering vector which remains nearly the same as it would be in the absence of interference. When $(I/N)_{max}$ becomes large, $\underline{k}^* \underline{d}$ given by (80) becomes small and a null develops in the direction of the interference. The factor

$$\cos^2(\mu_{Q_1^{-1}}) = \left|\underline{m}^* \underline{Q}_1^{-1} \underline{d}\right|^2 \Big/ (\underline{m}^* \underline{Q}_1^{-1} \underline{m})(\underline{d}^* \underline{Q}_1^{-1} \underline{d}) \tag{82}$$

appearing in (81) must be less than or equal to unity by the Schwarz inequality. It may be interpreted as the cosine squared of the generalized angle between $\underline{m}$ and $\underline{d}$ in the linear vector space in which length is defined relative to the metric $\underline{Q}_1^{-1}$, so that $\underline{m}^* \underline{Q}_1^{-1} \underline{m}$ is the length of $\underline{m}$.

The question of signal suppression arises when the steering vector $\underline{k}^*$ is made proportional to $\underline{m}^* \underline{R}^{-1}$ instead of $\underline{m}^* \underline{Q}^{-1}$ and $\underline{R}^{-1}$ contains the signal vector. It is of particular concern since the nulling embodied in the $\underline{R}^{-1}$ operation usually involves measured quantities and the matching operation involved in $\underline{m}^*$ usually is based on a priori assumptions about the signal direction vector.

Mathematically, signal suppression may be treated as a special case of interference rejection so that the results developed above may be applied directly. In particular, we

may look at the effect of the inclusion of signal in the estimator [4]

$$z = (\underline{m}^* \underline{R}^{-1} \underline{m})^{-1} \tag{83}$$

which is the power output of a beamformer with $\underline{k}^* = \underline{m}^* \underline{R}^{-1} / (\underline{m}^* \underline{R}^{-1} \underline{m})$. This $\underline{k}$ provides a minimum variance unbiased linear estimate of the signal [4, 7]. The quantity $z^{-1} = \underline{m}^* \underline{R}^{-1} \underline{m}$ may be obtained by replacing the initial factor $(1 + \sigma_d^2/\sigma_1^2)$ in (81) with $(1/\sigma_1^2)$ and redefining the other terms in (81) as follows:

$\underline{d}$: actual (measured) signal direction vector

$\underline{m}$: assumed (a priori) signal direction vector

$\underline{Q}_1$: noise only matrix

σ_1^2: input noise level

σ_d^2: input signal level

Mismatch occurs when $\underline{m} \neq \underline{d}$. The effect of mismatch can be seen by examining the ratio

$$\frac{(z)_{\underline{m} \neq \underline{d}}}{(z)_{\underline{m} = \underline{d}}} = \frac{\underline{d}^* \underline{Q}_1^{-1} \underline{d} \left[1 - \dfrac{1}{1 + (\underline{d}^* \underline{Q}_1^{-1} \underline{d} \; \sigma_d^2/\sigma_1^2)^{-1}} \right]}{\underline{m}^* \underline{Q}_1^{-1} \underline{m} \left[1 - \cos^2(\mu_{Q_1^{-1}}) \left(\dfrac{1}{1 + (\underline{d}^* \underline{Q}_1^{-1} \underline{d} \; \sigma_d^2/\sigma_1^2)^{-1}} \right) \right]} \tag{84}$$

There are two distinct effects of mismatch: First is the effect of the factor $(\underline{d}^* \underline{Q}_1^{-1} \underline{d} / \underline{m}^* \underline{Q}_1^{-1} \underline{m})$. This effect is the usual effect of mismatch discussed in Section III and has nothing to do with the inclusion of the signal in the matrix inversion. Second is what we shall call the anomalous signal suppression caused by the presence of $\cos^2(\mu_{Q_1^{-1}})$ in (84) which is a direct result of including the signal in the matrix inversion process. The anomalous signal suppression will be insignificant as long as the quantity $(\underline{d}^* \underline{Q}_1^{-1} \underline{d} \; \sigma_d^2/\sigma_1^2)$ is small, i.e., as long as the signal-to-noise ratio at the output of a perfectly matched optimum beamformer with $\underline{k}^* = \underline{d}^* \underline{Q}_1^{-1}$ would be small. If this weak signal criterion is not satisfied, the ratio given in (82) will be reduced through the inclusion of the signal in the matrix inversion and the mismatch between the actual and assumed signal direction vectors.

If we define $(S/N)_{max}$ as $(\underline{d}^* \underline{Q}_1^{-1} \underline{d} \; \sigma_d^2/\sigma_1^2)$ then (84) may be simplified to the following:

$$\frac{(z)_{\underline{m} \neq \underline{d}}}{(z)_{\underline{m} = d}} = \frac{\underline{d}^* \underline{Q}_1^{-1} \underline{d}}{\underline{m}^* \underline{Q}_1^{-1} \underline{m}} \left\{ \frac{1}{1 + (S/N)_{max} \sin^2(\mu_{Q_1^{-1}})} \right\} \tag{85}$$

where $\sin^2(\cdot) = 1 - \cos^2(\cdot)$. The factor $[1 + (S/N)_{max} \sin^2(\mu_{Q_1^{-1}})]^{-1}$ is a direct measure of the anomalous signal suppression.

V. OPTIMIZATION

There are a number of closely related optimization problems which have been or can be formulated. In this section we shall briefly sketch some of these problems with emphasis on formulations which somehow take sensitivity into consideration.

A. Beamformers

1. Maximum Gain and Minimum Variance

The problem of unconstrained array gain maximization is to choose $\underline{k}$ such that

$$G = \underline{k}^* \underline{P} \underline{k} / \underline{k}^* \underline{Q} \underline{k} \tag{7}$$

is maximized. The solution of this problem is known [6] to be choosing $\underline{k}$ proportional to the eigenvector corresponding to the largest eigenvalue of $(\underline{Q}^{-1}\underline{P})$. When $\underline{P} = \underline{m}\,\underline{m}^*$, the optimum $\underline{k}$ is proportional to $\underline{Q}\,\underline{m}$ which is well known [7, 10, 17] and easily shown by direct application of the Schwarz inequality [9].

The relationship between maximizing the gain and minimizing variance under a constraint on signal response lies in that maximizing G is equivalent to minimizing the denominator of (7) subject of a constraint on the numerator. Since the gain in (7) is invariant to a scaling of $\underline{k}$, the imposition of a constraint of the form $\underline{k}^*\underline{m} = 1$ has no effect on G and simply determines the constant of proportionality. Thus $\underline{k} = \underline{Q}^{-1}\underline{m} / (\underline{m}^*\underline{Q}^{-1}\underline{m})$ both maximizes G and satisfies the constraint $\underline{k}^*\underline{m} = 1$. The related problem of minimizing $\underline{k}^*\underline{R}\underline{k}$ subject to the same constraint leads to $\underline{k} = \underline{R}^{-1}\underline{m} / (\underline{m}^* \underline{R}^{-1}\underline{m})$.

2. Maximum Gain Subject to Sensitivity Constraint

As shown in Section III the generalized super-gain ratio ρ and the gain against white noise G_W play key roles in the sensitivity of beamformers when the perturbations are independent from sensor to sensor. The problem of maximizing the gain (7) subject to a constraint on

$$G_W = \underline{k}^* \underline{P} \underline{k} / \underline{k}^* \underline{k} \tag{86}$$

is most easily formulated as that of finding the $\underline{k}$ which minimizes $1/G$ with a constraint on $1/G_W$. That is minimizing

$$1/G + \lambda / G_W = \underline{k}^* (\underline{Q} + \lambda \underline{I}) \underline{k} \Big/ \underline{k}^* \underline{P} \underline{k} \tag{87}$$

where λ is a Lagrange multiplier. Finding the $\underline{k}$ which minimizes (87) is equivalent to finding the $\underline{k}$ which maximizes the reciprocal of (87), which in turn is an eigenvalue problem of the type discussed above. The optimum $\underline{k}$ is chosen to be proportional to the eigenvector corresponding to the largest eigenvalue of $[\underline{Q} + \lambda \underline{I}]^{-1} \underline{P}$. This problem was first addressed by Gilbert and Morgan [12] with $\underline{P} = \underline{m}\,\underline{m}^*$ so that the optimum $\underline{k}$ is equal to $[\underline{Q} + \lambda \underline{I}]^{-1} \underline{m}$.

A related problem is to maximize G subject to a constraint on ρ. In this case we maximize

$$G + \lambda \rho = \underline{k}^* [\underline{P} + \lambda \underline{I}] \underline{k} \Big/ \underline{k}^* \underline{Q} \underline{k} \tag{88}$$

Again we have a similar eigenvalue problem. Uzsoky and Solymar [24] examined this problem for the case of isotropic noise $\underline{Q} = \underline{Q}_o$. For the formulation of (88) to be consistent ρ must be specified within its allowable range given by (16). From (86) we see that the effect of adding the constraint on ρ is equivalent to that of adding a spatially white (spatially incoherent) component to the signal field.

Lo, Lee and Lee [15] review a number of optimization problems and present a numerical approach to the solution of the more difficult problem of maximizing G subject to a constraint on the Q-factor. They maximize

$$\left(\left| \underline{k}^* \underline{m} \right|^2 \Big/ \underline{k}^* \underline{Q}\, \underline{k} \right) + \left(\lambda\, \underline{k}^* \underline{k} \Big/ \underline{k}^* \underline{Q}_o \underline{k} \right)$$

All of these optimization problems are insensitive to a scaling of $\underline{k}$. Hence the addition of a linear constraint such as $\underline{k}^* \underline{m} = a \neq o$ is handled by simply scaling $\underline{k}$.

3. Maximization of Expected Quantities

A somewhat different approach to the problem of sensitivity is to take the type and anticipated size of perturbations into account before maximizing G. Thus perturbation terms are included in the ratio to be maximized. A special case of this approach has been taken by Cheng and Tseng [6] who maximize (71) and present numerical results for a linear endfire array of eight dipoles. Again the magnitude of $\underline{k}$ remains free so that an additional constraint of the form $\underline{k}^* \underline{m} = a \neq o$ can be handled by scaling $\underline{k}$.

4. Multiple Linear Constraints

In the preceding discussion a single linear constraint could usually be handled by using the degree of freedom of the magnitude of $\underline{k}$ which was left undetermined in the process of gain maximization. An exception arises when the constraint is a null of the form $\underline{k}^* \underline{d} = o$. This situation may be treated as a special case of the more general problem of

multiple linear constraints. Multiple constraints may be used to reduce sensitivity to signal perturbations by keeping the gain relatively constant over a range of signal perturbations. A similar technique may be used to control sidelobes in a specific neighborhood of directions. The problem can be formulated as that of minimizing the output power z subject to the constraint $\underline{H}^*\underline{k} = \underline{g}$. Each row vector of the constraint matrix $\underline{H}^*$ imposes a constraint of the form $\underline{h}_i^* \underline{k} = g_i$. Thus, the constraint matrix $\underline{H}^*$ has a row for each constraint. The total number of rows must be less than the number of sensors or the problem will be overspecified. Using a Lagrange multiplier vector $\underline{\lambda}^*$ we may minimize

$$z = \underline{k}^* \underline{R} \underline{k} + \underline{\lambda}^* [\underline{H}^* \underline{k} - \underline{g}] + [\underline{k}^* \underline{H} - \underline{g}^*] \underline{\lambda} \tag{89}$$

Completing the square [7] yields

$$z = [\underline{k}^* + \underline{\lambda}^* \underline{H}^* \underline{R}^{-1}] \underline{R} [\underline{R}^{-1} \underline{H} \underline{\lambda} + \underline{k}] - \underline{\lambda}^* \underline{H}^* \underline{R}^{-1} \underline{H} \underline{\lambda} - \underline{\lambda}^* \underline{g} - \underline{g}^* \underline{\lambda} \tag{90}$$

Since $\underline{k}$ appears only in the initial quadratic term of (90), the solution is obviously to make that term equal to zero by choosing

$$\underline{k} = -\underline{R}^{-1} \underline{H} \underline{\lambda} \tag{91}$$

Using the constraint $\underline{H}^*\underline{k} = \underline{g}$ to eliminate $\underline{\lambda}$, finally yields

$$\underline{k} = \underline{R}^{-1} \underline{H} [\underline{H}^* \underline{R}^{-1} \underline{H}]^{-1} \underline{g} \tag{92}$$

For this choice of $\underline{k}$ the output power $\underline{z}$ becomes

$$z = \underline{g}^* [\underline{H}^* \underline{R}^{-1} \underline{H}]^{-1} \underline{g} \tag{93}$$

The use of an estimate for $\underline{R}^{-1}$ in (91) is a generalization of the estimator (83). One approach to maintaining signal response is to have each row vector of $\underline{H}^*$ be a steering vector in the neighborhood of the steering direction $\underline{m}$ and to let $\underline{g}$ be a vector with the number one as each component.

B. General Array Processor

So far our discussion has centered on beamformers which have the structure of Figure 1. A more general array processor structure is illustrated in Figure 2. In the more general processor there are multiple outputs obtained by a matrix filtering of the input signals. In the processor of Figure 2, $\underline{K}^*$ is a matrix. While the processor of Figure 1 could be followed by a simple square-law detector and averager, the matrix filter $\underline{K}^*$ may be followed by a more general quadratic processor. It is known [7, 18] that the structure of Figure 1 is only optimum when $\underline{P}$ is a simple dyadic. Since the effects of perturbation is to destroy the dyadic nature of the expectation of $\underline{P}$ we are naturally led to the consideration of more the general processor.

For example, suppose that the signal and noise were both completely incoherent from sensor to sensor, i.e., $\underline{P} = \underline{Q} = \underline{I}$. Then all beamformers of the type of Figure 1 will provide no gain (G = 1) while an incoherent combination of the sensor outputs can provide a gain of $\sqrt{M}$.

A processor which forms multiple closely spaced beams and averages output power across these beams is a special case of the general array processor.

In the general array processor the output power is defined as

$$z = \text{trace}\,(\underline{K}^* \underline{R}\,\underline{K}) = E(y) \tag{94}$$

In order to be able to properly account for the potential of incoherent gain, we define a gain in terms of the detection index at the output of a general quadratic processor [7].

$$G = \frac{E(y|S+N) - E(y|N)}{\left\{E(y^2|N) - E^2(y|N)\right\}^{1/2}} \left(\frac{\sigma_n^2}{\sigma_s^2}\right) \tag{95}$$

When the noise is Gaussian [7], (95) may be written as

$$G = \text{trace}\,[\underline{K}^* \underline{P}\,\underline{K}] \Big/ \{\text{trace}\,[(\underline{K}^* \underline{Q}\,\underline{K})^2]\}^{1/2} \tag{96}$$

Notice that (96) reduces to (7) when $\underline{K}$ is a column vector.

The problem of maximizing (96) has been solved [7, 9] by direct application of the following Schwarz inequality:

$$\left|\text{trace}\,(\underline{A}^* \underline{B})\right|^2 \leq \text{trace}\,(\underline{A}^* \underline{A})\ \text{trace}\,(\underline{B}^* \underline{B}) \tag{97}$$

The optimum choice of $\underline{K}$ is

$$\underline{K} = c\,\underline{Q}^{-1}\underline{A} \tag{98}$$

where c is an arbitrary scalar constant of proportionality and $\underline{P} = \underline{A}\underline{A}^*$. The matrix $\underline{A}$, assumed for convenience to be of full rank, will have M rows and r columns where r is the rank of the matrix $\underline{P}$. The maximum value of gain (96) is

$$G = \{\text{trace}\,[(\underline{P}\underline{Q}^{-1})^2]\}^{1/2} \tag{99}$$

This maximum gain cannot be achieved by any $\underline{K}$ with less than r columns.

The problem of maximizing gain (96) determines $\underline{K}$ to within a scalar multiple. Thus, constraints of the form trace $(\underline{H}^* \underline{K}) = a \neq o$ may be handled by a simple scaling as before.

The problem of minimizing trace $(\underline{K}^* \underline{R}\,\underline{K})$ subject to multiple linear constraints of the form $\underline{H}^* \underline{K} = \underline{L}$ is readily handled by completing the square similar to the way it was done in (89). For details, see the development in equations (28) to (31) of [7]. The solution is

$$\underline{K} = \underline{R}^{-1}\underline{H}\,[\underline{H}^* \underline{R}^{-1}\underline{H}]^{-1}\,\underline{L} \tag{100}$$

for this value of $\underline{K}$,

$$\underline{K}^* \underline{R} \underline{K} = \underline{L}^* [\underline{H}^* \underline{R}^{-1} \underline{H}]^{-1} \underline{L} \qquad (101)$$

Equations (100) and (101) are generalizations of (92) and (93). A further generalization of the estimator (83) is therefore

$$z = \text{trace} \left\{ \underline{L}^* [\underline{H}^* \underline{R}^{-1} \underline{H}]^{-1} L \right\} \qquad (102)$$

C. Implementation

Most of the work [5, 13, 21, 26] in adaptive beamforming has dealt with unconstrained optimization problems so that there remains much unbroken ground in the field of constrained optimization.

Some exceptions do exist. The work in antenna array optimization [6, 12, 22, 24] has been concerned with sensitivities but not concerned with appropriate algorithms for on-line adaptation. The problem of multiple linear constraints leads naturally to stochastic versions of gradient-projection type algorithms [11, 19]. This approach has also been suggested [27] for non-linear constraints. However, slow convergence may be anticipated.

VI. CONCLUSION

General sensitivity measures have been developed for the cases of perturbations to the signal field, the noise field and the steering vector. These sensitivity measures may be used to test the practicality of particular processors. It was found that the white noise gain and the generalized super-gain ratio are key parameters in determining the various sensitivities when the perturbations are independent from sensor to sensor.

The problem of anomalous signal suppression through the inclusion of the signal in the matrix inversion and subsequent mismatch has been treated as a special case of interference rejection. A simple expression for this anomalous signal suppression has been presented. This signal suppression can only be significant if the signal-to-noise ratio would be large at the output of an optimum beamformer steered perfectly in the signal direction.

Various beamformer optimization problems have been considered. Constraining ρ was found to be equivalent to adding a spatially incoherent component to the signal field prior to unconstrained optimization. Constraining G_W was found to be equivalent to adding a spatially white component to the noise field.

In many important situations the simple beamformer structure is not optimum. A more general array processing configuration has been presented which provides for

incoherent combination of the outputs of a number of simple beamformers. Optimization problems associated with this processor have been solved providing generalization of the results for simple beamformers.

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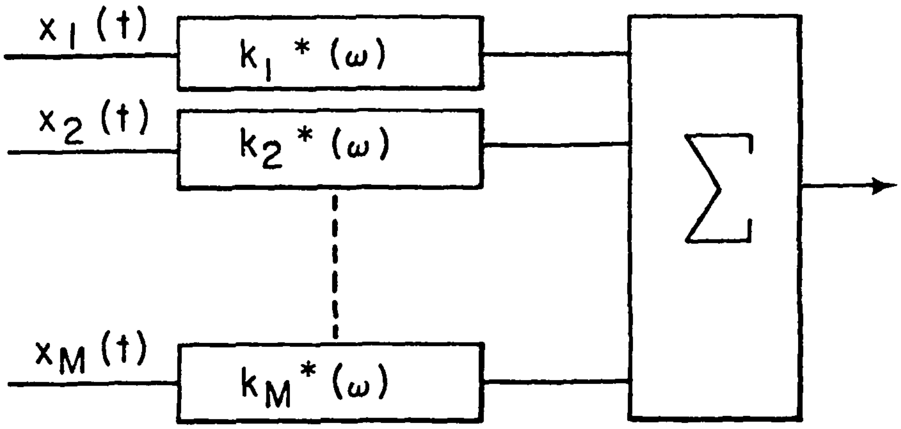

Figure 1. General Beamformer Configuration.

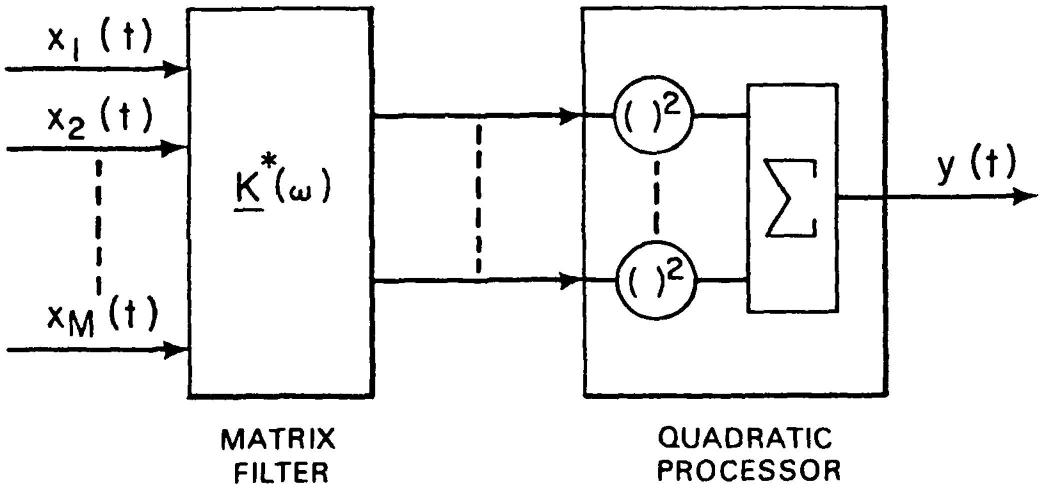

Figure 2. General Array Processor.